CES 6116 FEM-Review
Matrix Structural Analysis

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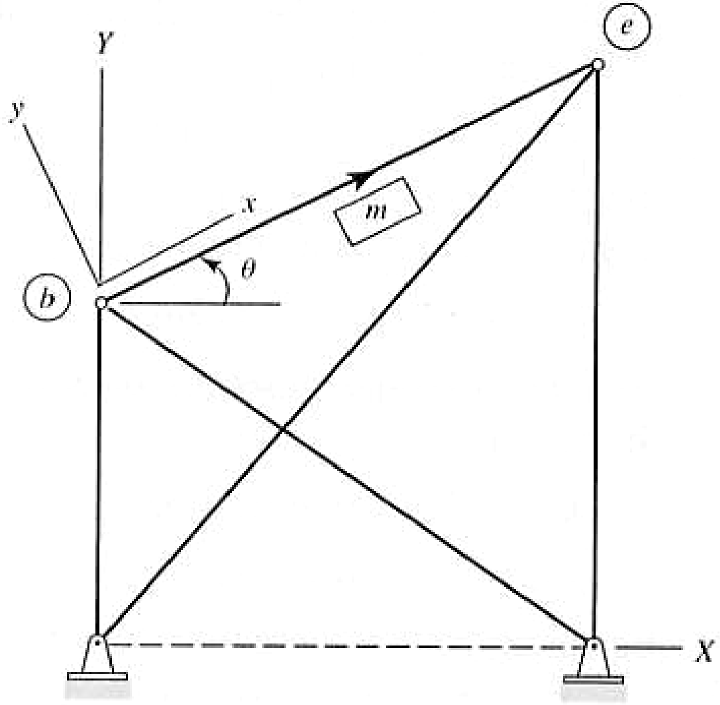
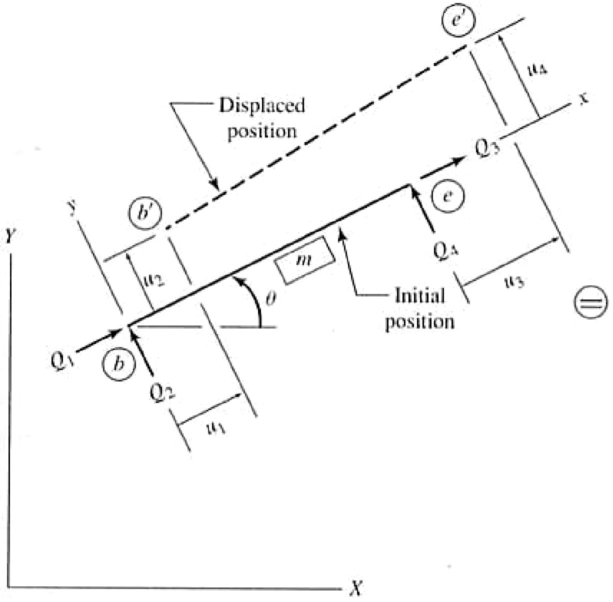
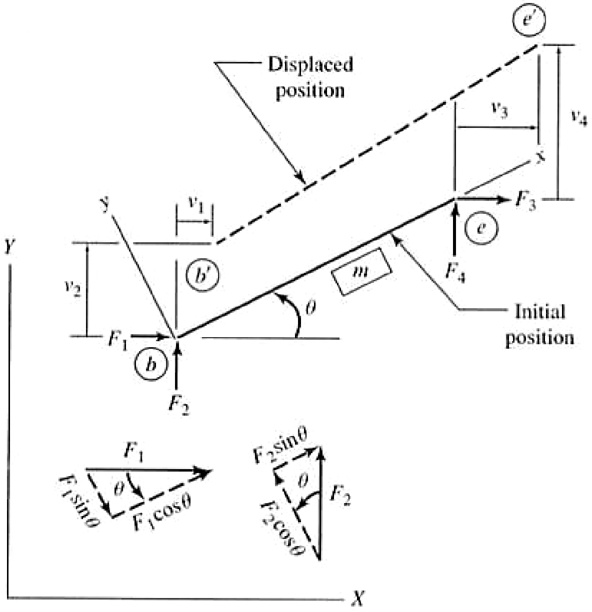
Coordinate Transformation-Truss Elements



(a) Truss

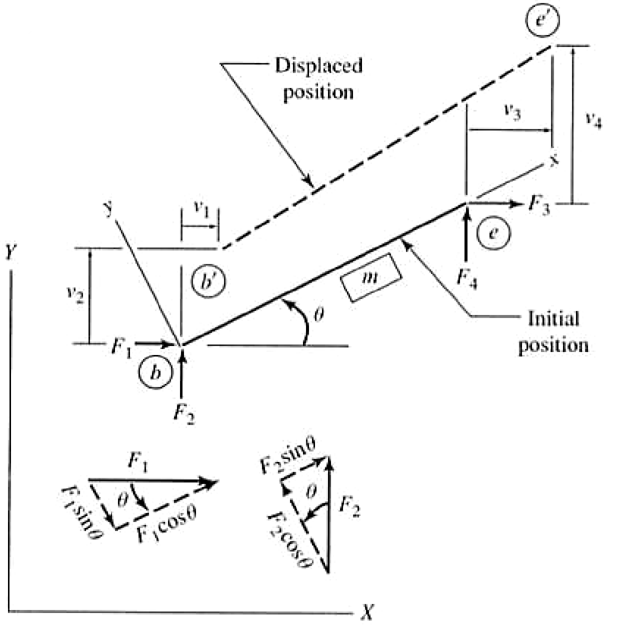


(b) Member End Forces and End Displacements in the Local Coordinate System



(c) Member End Forces and End Displacements in the Global Coordinate System

Global to Local Coordinate Transformation



(c) Member End Forces and End Displacements in the Global Coordinate System

In Local Axes

In Global Axes

$$Q_1 = F_1 \cos\theta + F_2 \sin\theta$$

$$Q_2 = -F_1 \sin\theta + F_2 \cos\theta$$

$$Q_3 = F_3 \cos\theta + F_4 \sin\theta$$

$$Q_4 = -F_3 \sin\theta + F_4 \cos\theta$$

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}$$

Transformation Matrix

$$\boxed{\mathbf{Q} = \mathbf{T}\mathbf{F}}$$

$$\boxed{\mathbf{u} = \mathbf{T}\mathbf{v}}$$

$$\mathbf{T} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$$\cos\theta = \frac{X_e - X_b}{L} = \frac{X_e - X_b}{\sqrt{(X_e - X_b)^2 + (Y_e - Y_b)^2}}$$

$$\sin\theta = \frac{Y_e - Y_b}{L} = \frac{Y_e - Y_b}{\sqrt{(X_e - X_b)^2 + (Y_e - Y_b)^2}}$$

Member Angles

$$\cos\theta = \frac{X_e - X_b}{L} = \frac{X_e - X_b}{\sqrt{(X_e - X_b)^2 + (Y_e - Y_b)^2}}$$

$$\sin\theta = \frac{Y_e - Y_b}{L} = \frac{Y_e - Y_b}{\sqrt{(X_e - X_b)^2 + (Y_e - Y_b)^2}}$$

Local to Global Transformation

$$F_1 = Q_1 \cos\theta - Q_2 \sin\theta \qquad F_3 = Q_3 \cos\theta - Q_4 \sin\theta$$

$$F_2 = Q_1 \sin\theta + Q_2 \cos\theta \qquad F_4 = Q_3 \sin\theta + Q_4 \cos\theta$$

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix}$$

Transformation Matrix

Global to Local

$$\boxed{\mathbf{Q} = \mathbf{T}\mathbf{F}}$$

$$\boxed{\mathbf{u} = \mathbf{T}\mathbf{v}}$$

Local to Global

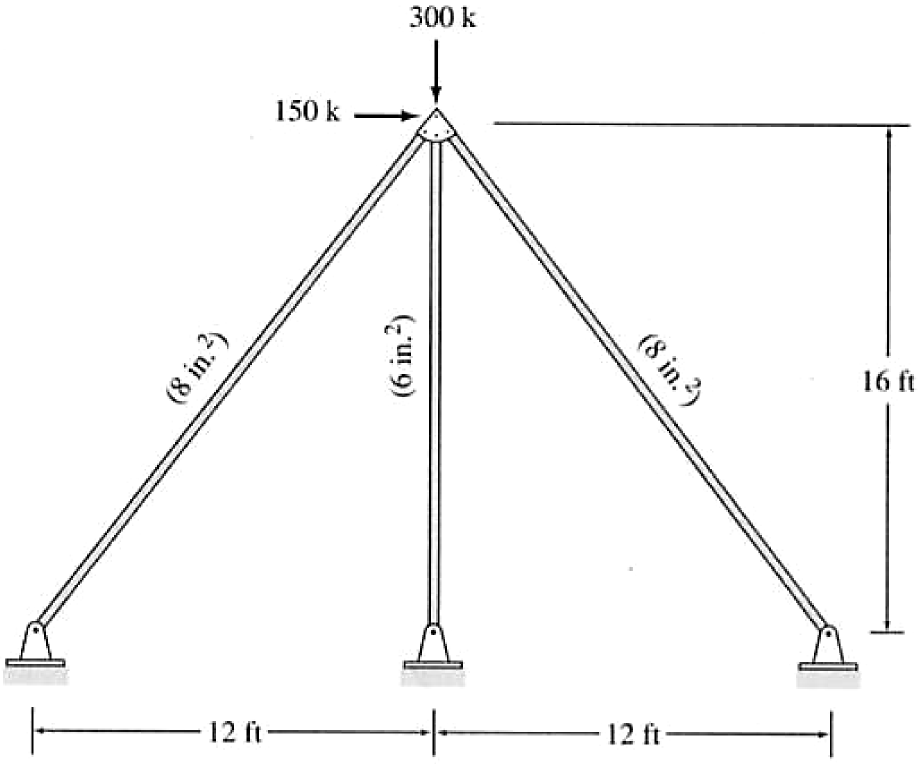
$$\boxed{\mathbf{F} = \mathbf{T}^T \mathbf{Q}}$$

$$\boxed{\mathbf{v}=\mathbf{T}^T\mathbf{u}}$$

$\mathbf{T}^{-1} = \mathbf{T}^T$ → *Orthogonal Matrix*

Truss Analysis - Example

- Determine the joint displacements, member axial deformations, suppor reactions of the truss

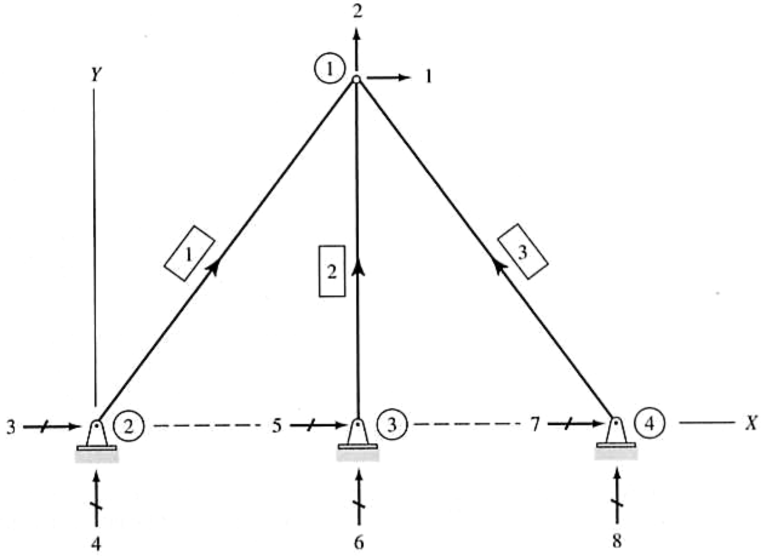


E = 29,000 ksi

(a) Truss

Truss Analysis - Example

- Member Stiffness Matrices



(b) Analytical Model

$$\mathbf{K}_1 = \frac{(29{,}000)(8)}{(20)(12)}\begin{bmatrix} 0.36 & 0.48 & -0.36 & -0.48 \\ 0.48 & 0.64 & -0.48 & -0.64 \\ -0.36 & -0.48 & 0.36 & 0.48 \\ -0.48 & -0.64 & 0.48 & 0.64 \end{bmatrix}$$

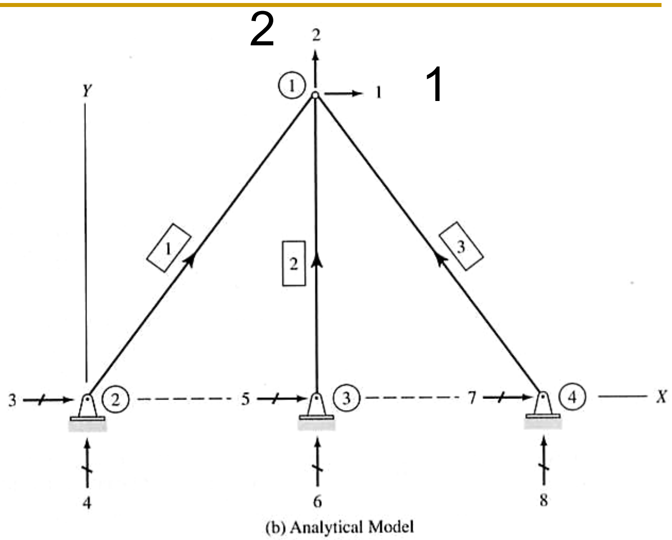
$$\mathbf{K}_2 = \begin{array}{c} \begin{array}{cccc} 5 & 6 & 1 & 2 \end{array} \\ \left[\begin{array}{cc:cc} 0 & 0 & 0 & 0 \\ 0 & 906.25 & 0 & -906.25 \\ \hdashline 0 & 0 & 0 & 0 \\ 0 & -906.25 & 0 & 906.25 \end{array}\right] \end{array} \begin{array}{c} \\ 5 \\ 6 \\ 1 \\ 2 \end{array} \text{ k/in.}$$

$$\mathbf{K}_3 = \begin{array}{c} \begin{array}{cccc} 7 & 8 & 1 & 2 \end{array} \\ \left[\begin{array}{cc:cc} 348 & -464 & -348 & 464 \\ -464 & 618.67 & 464 & -618.67 \\ \hdashline -348 & 464 & 348 & -464 \\ 464 & -618.67 & -464 & 618.67 \end{array}\right] \end{array} \begin{array}{c} \\ 7 \\ 8 \\ 1 \\ 2 \end{array} \text{ k/in.}$$

$$\mathbf{S} = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} (348 + 0 + 348) & (464 + 0 - 464) \\ (464 + 0 - 464) & (618.67 + 906.25 + 618.67) \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array} = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 696 & 0 \\ 0 & 2{,}143.6 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array} \text{ k/in.}$$

(c) Structure Stiffness Matrix

Truss Analysis - Example



(b) Analytical Model

$$\mathbf{S} = \begin{array}{cc|c} 1 & 2 & \\ (348 + 0 + 348) & (464 + 0 - 464) & 1 \\ (464 + 0 - 464) & (618.67 + 906.25 + 618.67) & 2 \end{array} = \begin{bmatrix} 696 & 0 \\ 0 & 2{,}143.6 \end{bmatrix}\begin{matrix}1\\2\end{matrix} \text{ k/in.}$$

(c) Structure Stiffness Matrix

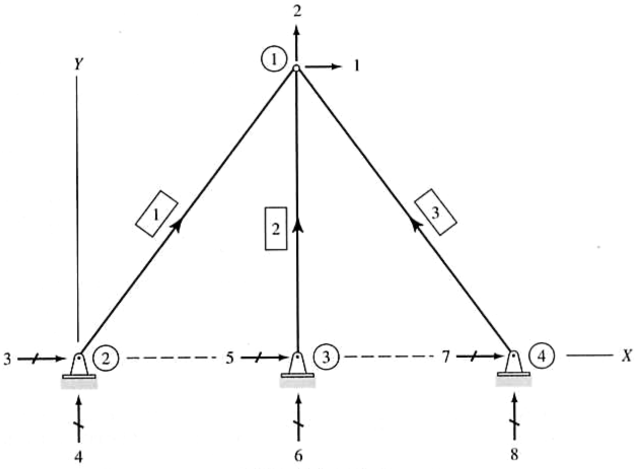
{P}=[S]{d}

$$\mathbf{P} = \begin{bmatrix} 150 \\ -300 \end{bmatrix} \text{k}$$

$$\begin{bmatrix} 150 \\ -300 \end{bmatrix} = \begin{bmatrix} 696 & 0 \\ 0 & 2,143.6 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$\mathbf{d} = \begin{bmatrix} 0.21552 \\ -0.13995 \end{bmatrix} \text{ in.} \qquad d_1 = 0.21552 \text{ in.} \qquad d_2 = -0.13995 \text{ in.}$$

Truss Analysis - Example



(b) Analytical Model

MEMBER 1

Member 1 Deformations in Global CS.

$$\mathbf{d} = \begin{bmatrix} 0.21552 \\ -0.13995 \end{bmatrix} \text{ in.}$$

$$\mathbf{v}_1 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \begin{matrix} 3 \\ 4 \\ 1 \\ 2 \end{matrix} = \begin{bmatrix} 0 \\ 0 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.21552 \\ -0.13995 \end{bmatrix} \text{ in.}$$

$$\mathbf{T}_1 = \begin{bmatrix} 0.6 & 0.8 & 0 & 0 \\ -0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0.8 \\ 0 & 0 & -0.8 & 0.6 \end{bmatrix}$$

$$\mathbf{u}_1 = \mathbf{T}_1 \mathbf{v}_1$$

$$\mathbf{Q}_1 = \mathbf{k}_1 \mathbf{u}_1$$

Stiffness and Member Forces in Local CS

$$\mathbf{k}_1 = \begin{bmatrix} 966.67 & 0 & -966.67 & 0 \\ 0 & 0 & 0 & 0 \\ -966.67 & 0 & 966.67 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{Q}_1 = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} 966.67 & 0 & -966.67 & 0 \\ 0 & 0 & 0 & 0 \\ -966.67 & 0 & 966.67 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.017352 \\ -0.25639 \end{bmatrix} = \begin{bmatrix} -16.774 \\ 0 \\ 16.774 \\ 0 \end{bmatrix}$$

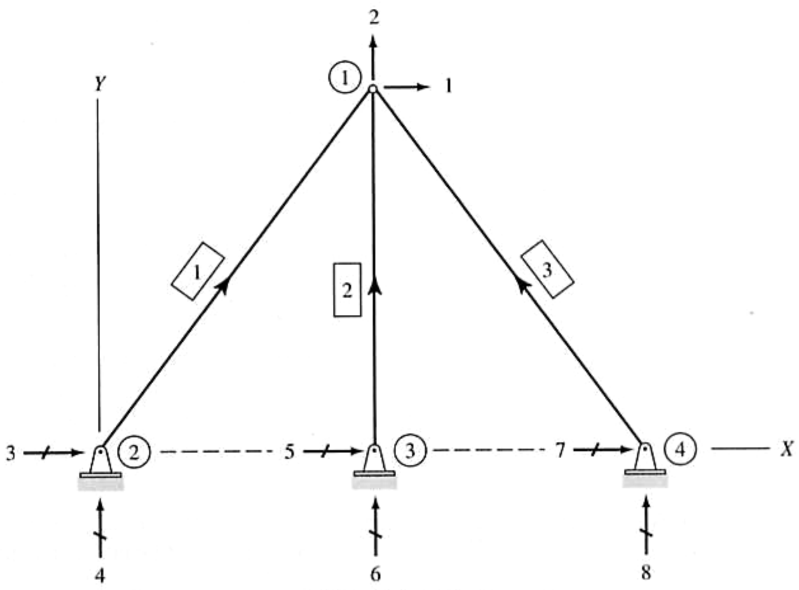
Truss Analysis - Example

MEMBER 1

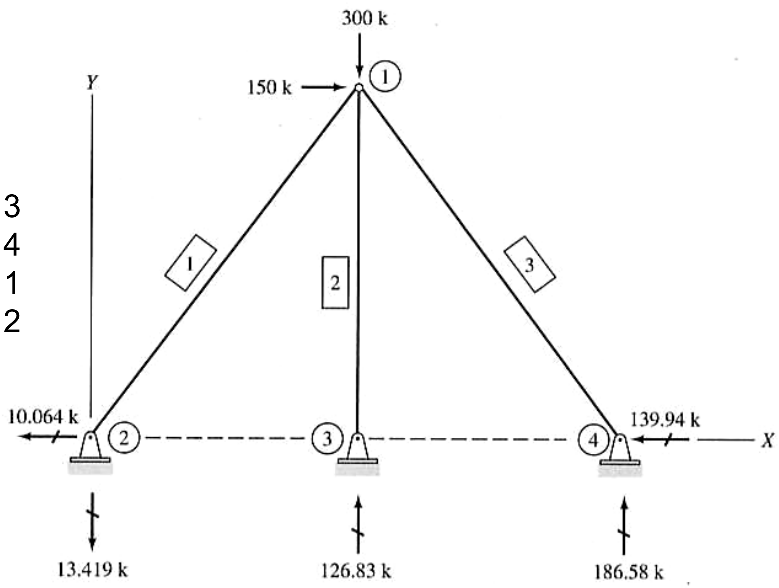
Member 1 Force Transformation from Local to Global

$$F = T^T Q$$

$$\mathbf{F}_1 = \begin{bmatrix}F_1\\F_2\\F_3\\F_4\end{bmatrix} = \begin{bmatrix}0.6 & -0.8 & 0 & 0\\0.8 & 0.6 & 0 & 0\\0 & 0 & 0.6 & -0.8\\0 & 0 & 0.8 & 0.6\end{bmatrix}\begin{bmatrix}-16.774\\0\\16.774\\0\end{bmatrix} = \left[\begin{array}{c}-10.064\\-13.419\\\hline 10.064\\13.419\end{array}\right]\begin{array}{c}3\\4\\1\\2\end{array}$$



(b) Analytical Model



(f) Support Reactions

Truss Analysis-Example

MEMBER 2

Member 2 Deform.s in Global CS.

$$\mathbf{u_2=T_2v_2}$$

$$\mathbf{v}_2 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \begin{matrix} 5 \\ 6 \\ 1 \\ 2 \end{matrix} = \begin{bmatrix} 0 \\ 0 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.21552 \\ -0.13995 \end{bmatrix}$$

$$\mathbf{u}_2 = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.21552 \\ -0.13995 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -0.13995 \\ -0.21552 \end{bmatrix}$$

$$\mathbf{Q_2=k_2u_2}$$

$$\mathbf{Q}_2 = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = 906.25 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -0.13995 \\ -0.21552 \end{bmatrix} = \begin{bmatrix} 126.83 \\ 0 \\ -126.83 \\ 0 \end{bmatrix}$$

$$\mathrm{F=T^TQ}$$

$$\mathbf{F}_2 = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 126.83 \\ 0 \\ -126.83 \\ 0 \end{bmatrix} = \left[\begin{array}{c} 0 \\ 126.83 \\ \hdashline 0 \\ -126.83 \end{array}\right] \begin{matrix} 5 \\ 6 \\ 1 \\ 2 \end{matrix}$$

Truss Analysis-Example

MEMBER 3

Member 3 Deformations in Global CS.

$$\mathbf{v}_3 = \begin{array}{c} \left[\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\right] \end{array} \begin{array}{c} 7 \\ 8 \\ 1 \\ 2 \end{array} = \begin{bmatrix} 0 \\ 0 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.21552 \\ -0.13995 \end{bmatrix}$$

$\mathbf{u_3 = T_3 v_3}$

$\mathbf{F_3 = K_3 v_3}$

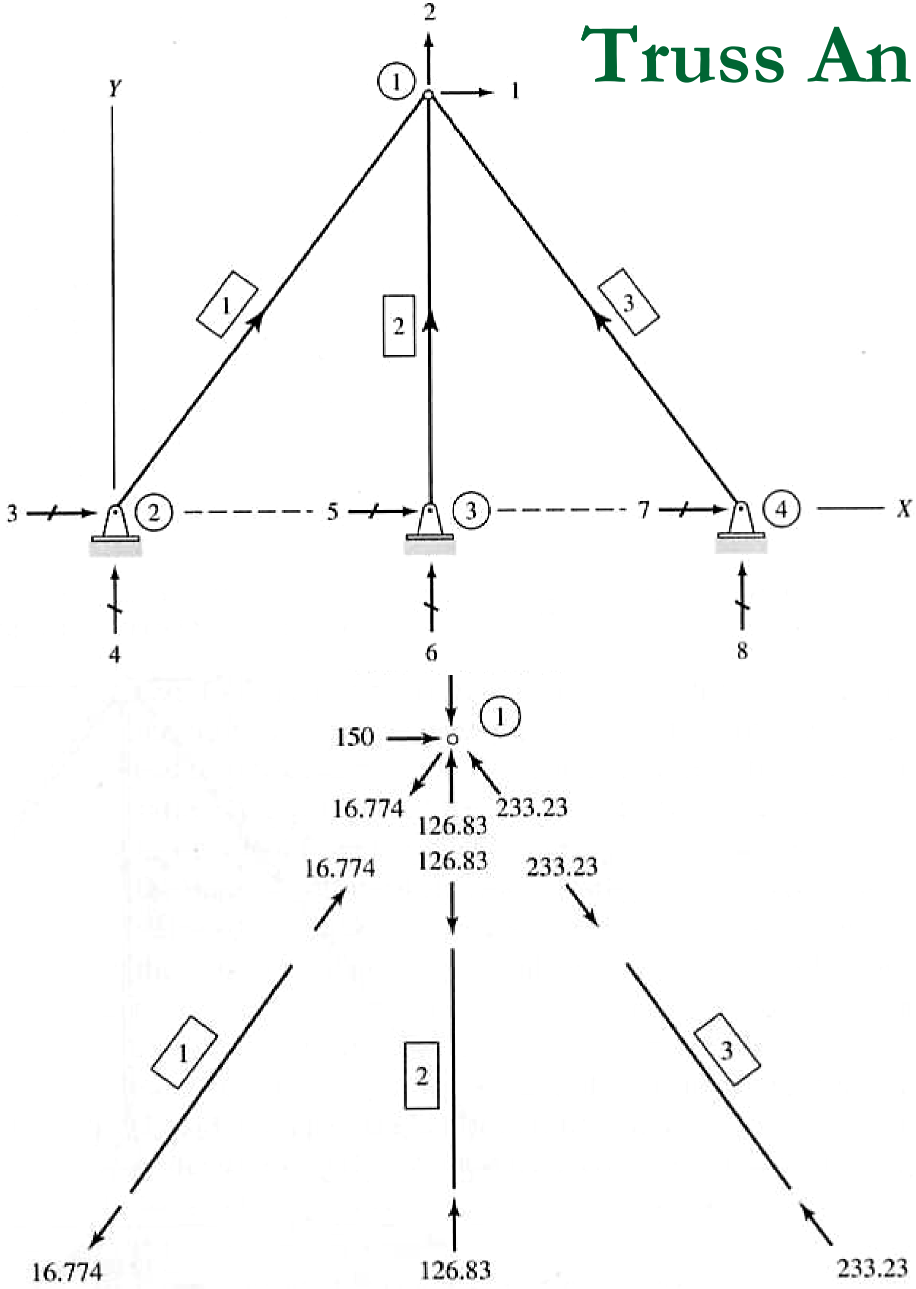
$$\mathbf{F}_3 = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} 348 & -464 & -348 & 464 \\ -464 & 618.67 & 464 & -618.67 \\ -348 & 464 & 348 & -464 \\ 464 & -618.67 & -464 & 618.67 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.21552 \\ -0.13995 \end{bmatrix}$$

$$= \left[\begin{array}{c} -139.94 \\ 186.58 \\ \hdashline 139.94 \\ -186.58 \end{array}\right] \begin{array}{c} 7 \\ 8 \\ 1 \\ 2 \end{array} \text{ k}$$

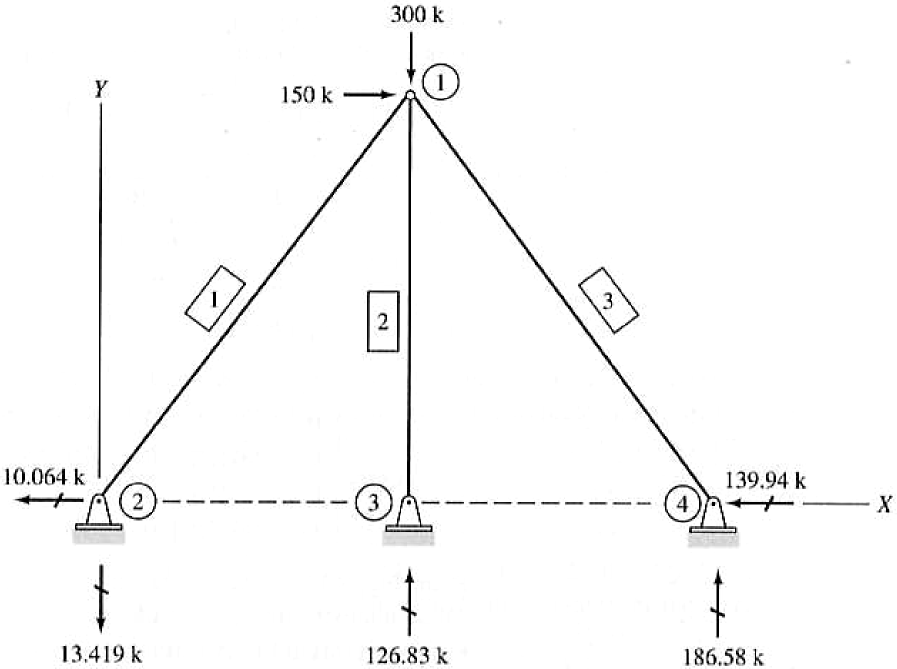
$\{Q\}=[T]\{F\}$

$$\mathbf{Q}_3 = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} -0.6 & 0.8 & 0 & 0 \\ -0.8 & -0.6 & 0 & 0 \\ 0 & 0 & -0.6 & 0.8 \\ 0 & 0 & -0.8 & -0.6 \end{bmatrix} \begin{bmatrix} -139.94 \\ 186.58 \\ 139.94 \\ -186.58 \end{bmatrix} = \begin{bmatrix} 233.23 \\ 0 \\ -233.23 \\ 0 \end{bmatrix}$$

Truss Analysis-Example



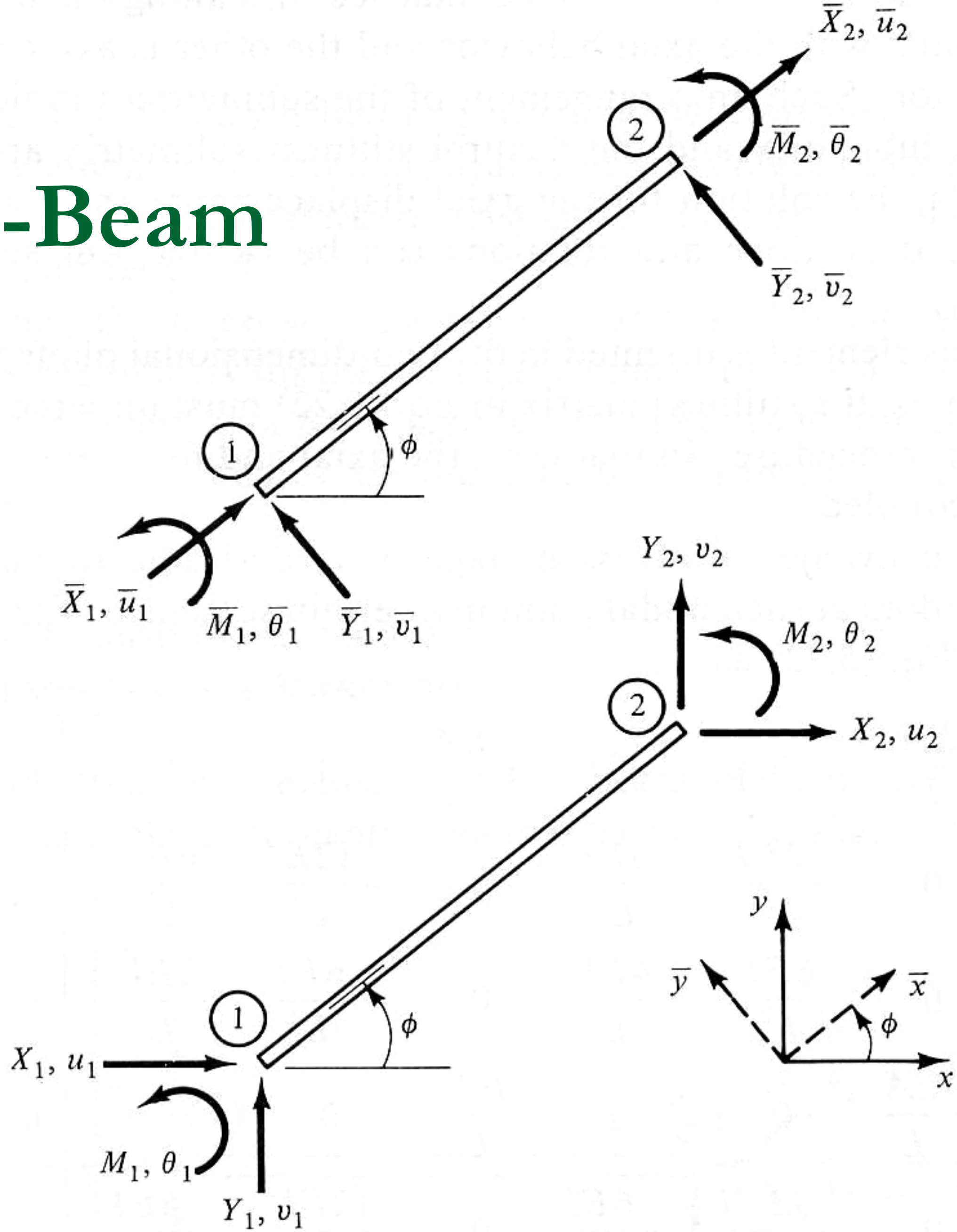
$$\mathbf{R} = \begin{array}{c} \left[\begin{array}{c} -10.064 \\ -13.419 \\ 0 \\ 126.83 \\ -139.94 \\ 186.58 \end{array}\right] \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \end{array} \text{ k}$$



Coordinate Transformation-Beam Elements

Local Coordinates

Global Coordinates



Coordinate Transformation

$$\begin{array}{ccccc} \textbf{Local Coordinates} & & \textbf{Transformation Matrix} & & \textbf{Global Coordinates} \\ \left\{\begin{array}{c} \bar{X}_1 \\ \bar{Y}_1 \\ \bar{M}_1 \\ \hline \bar{X}_2 \\ \bar{Y}_2 \\ \bar{M}_2 \end{array}\right\} & = & \left[\begin{array}{ccc|ccc} \lambda & \mu & 0 & 0 & 0 & 0 \\ -\mu & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda & \mu & 0 \\ 0 & 0 & 0 & -\mu & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] & & \left\{\begin{array}{c} X_1 \\ Y_1 \\ M_1 \\ \hline X_2 \\ Y_2 \\ M_2 \end{array}\right\} \end{array}$$

$$\{\bar{\mathbf{F}}\} = [\mathbf{T}]\{\mathbf{F}\}$$

$$\boxed{\begin{array}{l} \lambda = \cos\phi \\ \mu = \sin\phi \end{array}}$$

Coordinate Transformation-Derivations

[T] is orthogonal matrix

$$[\mathbf{T}]^{-1} = [\mathbf{T}]^{T}$$

We can write {F} as follows:

$$\{\mathbf{F}\} = [\mathbf{T}]^{-1}\{\bar{\mathbf{F}}\} = [\mathbf{T}]^{T}\{\bar{\mathbf{F}}\}$$

Also we can write the following:

$$\{\bar{\mathbf{q}}\} = [\mathbf{T}]\{\mathbf{q}\}$$

Substituting, we obtain,

$$[\mathbf{T}]\{\mathbf{F}\} = [\bar{\mathbf{k}}][\mathbf{T}]\{\mathbf{q}\}$$

Finally,

$$\boxed{\{\mathbf{F}\} = [\mathbf{T}]^{T}[\bar{\mathbf{k}}][\mathbf{T}]\{\mathbf{q}\}}$$

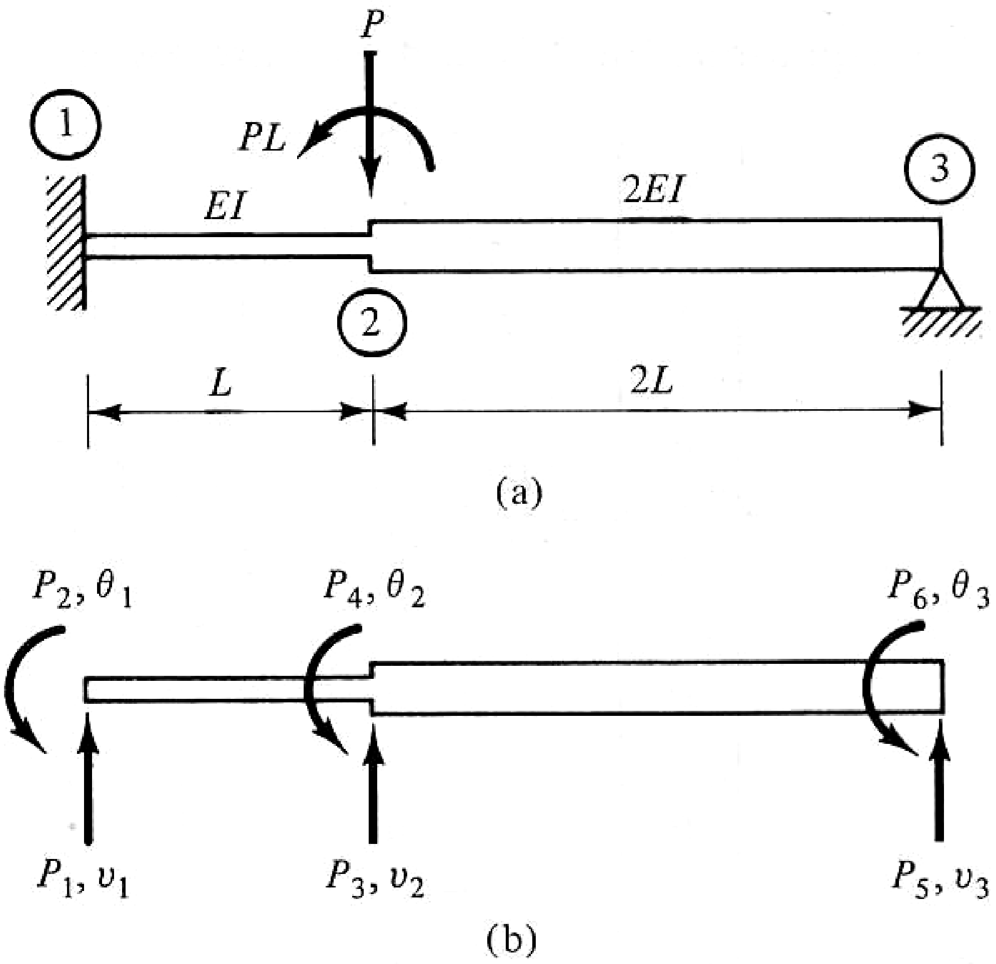
Stiffness Matrix for a Plane Frame Element Oriented at an Angle ϕ

$$\boxed{[\mathbf{T}]^T[\bar{\mathbf{k}}][\mathbf{T}]}$$

$$\begin{Bmatrix} X_1 \\ Y_1 \\ M_1 \\ X_2 \\ Y_2 \\ M_2 \end{Bmatrix} = \frac{EI}{L}\begin{bmatrix} R\lambda^2 + \frac{12}{L^2}\mu^2 & & & & & \\ \left(R - \frac{12}{L^2}\right)\lambda\mu & R\mu^2 + \frac{12}{L^2}\lambda^2 & & & \text{symmetric} & \\ -\frac{6}{L}\mu & \frac{6}{L}\lambda & 4 & & & \\ -R\lambda^2 - \frac{12}{L^2}\mu^2 & \left(-R + \frac{12}{L^2}\right)\lambda\mu & \frac{6}{L}\mu & R\lambda^2 + \frac{12}{L^2}\mu^2 & & \\ \left(-R + \frac{12}{L^2}\right)\lambda\mu & -R\mu^2 - \frac{12}{L^2}\lambda^2 & -\frac{6}{L}\lambda & \left(R - \frac{12}{L^2}\right)\lambda\mu & R\mu^2 + \frac{12}{L^2}\lambda^2 & \\ -\frac{6}{L}\mu & \frac{6}{L}\lambda & 2 & \frac{6}{L}\mu & -\frac{6}{L}\lambda & 4 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$\boxed{R = A/I}$$

Example Problem #1



(a)

(b)

Example Problem #1 (cont.)

$$\begin{Bmatrix} Y_1 \\ M_1 \\ Y_2 \\ M_2 \end{Bmatrix} = \frac{EI}{L} \begin{bmatrix} \frac{12}{L^2} & \text{symmetric} & & \\ \frac{6}{L} & 4 & & \\ -\frac{12}{L^2} & -\frac{6}{L} & \frac{12}{L^2} & \\ \frac{6}{L} & 2 & -\frac{6}{L} & 4 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

For Element 1-2

$$\begin{Bmatrix} Y_2 \\ M_2 \\ Y_3 \\ M_3 \end{Bmatrix} = \frac{EI}{L} \begin{bmatrix} \frac{3}{L^2} & \text{symmetric} & & \\ \frac{3}{L} & 4 & & \\ -\frac{3}{L^2} & -\frac{3}{L} & \frac{3}{L^2} & \\ \frac{3}{L} & 2 & -\frac{3}{L} & 4 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix}$$

For Element 2-3

Example Problem #1 (cont.)

$$P_1 = Y_1$$

$$P_2 = M_1$$

$$P_3 = Y_2 \text{ of element 1-2} + Y_2 \text{ of element 2-3}$$

$$P_4 = M_2 \text{ of element 1-2} + M_2 \text{ of element 2-3}$$

$$P_5 = Y_3$$

$$P_6 = M_3$$

Example Problem #1 (cont.)

Combining Element 1-2 and Element 1-2 Stiffness Matrices

$$
\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix}
= \frac{EI}{L}
\begin{bmatrix}
\frac{12}{L^2} & & & & & \\
\frac{6}{L} & 4 & & \text{symmetric} & & \\
-\frac{12}{L^2} & -\frac{6}{L} & \frac{12}{L^2}+\frac{3}{L^2} & & & \\
\frac{6}{L} & 2 & -\frac{6}{L}+\frac{3}{L} & 4+4 & & \\
0 & 0 & -\frac{3}{L^2} & -\frac{3}{L} & \frac{3}{L^2} & \\
0 & 0 & \frac{3}{L} & 2 & -\frac{3}{L} & 4
\end{bmatrix}
\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix}
$$

Example Problem #1 (cont.)

$v_1 = \theta_1 = v_3 = 0$ ***Boundary Conditions***

$P_1 =$ unknown reaction force at point 1

$P_2 =$ unknown reaction moment at point 1

$P_3 = -P$ (opposite to the positive y-direction)

$P_4 = PL$

$P_5 =$ unknown reaction force at point 3

$P_6 = 0$

Example Problem #1 (cont.)

Re-arranging based on BCs and known loadings, we get

$$\begin{Bmatrix} P_3 \\ P_4 \\ P_6 \\ P_1 \\ P_2 \\ P_5 \end{Bmatrix} = \frac{EI}{L} \begin{bmatrix} -\frac{12}{L^2} & -\frac{6}{L} & \frac{15}{L^2} & -\frac{3}{L} & -\frac{3}{L^2} & \frac{3}{L} \\ \frac{6}{L} & 2 & -\frac{3}{L} & 8 & -\frac{3}{L} & 2 \\ 0 & 0 & \frac{3}{L} & 2 & -\frac{3}{L} & 4 \\ \frac{12}{L^2} & \frac{6}{L} & -\frac{12}{L^2} & \frac{6}{L} & 0 & 0 \\ \frac{6}{L} & 4 & -\frac{6}{L} & 2 & 0 & 0 \\ 0 & 0 & -\frac{3}{L^2} & -\frac{3}{L} & \frac{3}{L^2} & -\frac{3}{L} \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix}$$

Example Problem #1 (cont.)

Writing knowns and unknowns, we get 6 eqns for 6 unknowns:

$$\left\{\begin{array}{c} P_3 = -P \\ P_4 = PL \\ P_6 = 0 \\ \hline P_1 = ? \\ P_2 = ? \\ P_5 = ? \end{array}\right\} = \frac{EI}{L}\left[\begin{array}{ccc|ccc} \frac{15}{L^2} & -\frac{3}{L} & \frac{3}{L} & -\frac{12}{L^2} & -\frac{6}{L} & -\frac{3}{L^2} \\ -\frac{3}{L} & 8 & 2 & \frac{6}{L} & 2 & -\frac{3}{L} \\ \frac{3}{L} & 2 & 4 & 0 & 0 & -\frac{3}{L} \\ \hline -\frac{12}{L^2} & \frac{6}{L} & 0 & \frac{12}{L^2} & \frac{6}{L} & 0 \\ -\frac{6}{L} & 2 & 0 & \frac{6}{L} & 4 & 0 \\ -\frac{3}{L^2} & -\frac{3}{L} & -\frac{3}{L} & 0 & 0 & \frac{3}{L^2} \end{array}\right]\left\{\begin{array}{c} v_2 = ? \\ \theta_2 = ? \\ \theta_3 = ? \\ \hline v_1 = 0 \\ \theta_1 = 0 \\ v_3 = 0 \end{array}\right\}$$

Example Problem #1 (cont.)

$$\begin{Bmatrix} -P \\ PL \\ 0 \end{Bmatrix} = \frac{EI}{L} \begin{bmatrix} \frac{15}{L^2} & -\frac{3}{L} & \frac{3}{L} \\ -\frac{3}{L} & 8 & 2 \\ \frac{3}{L} & 2 & 4 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

Solve for unknown disp.

$$\begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \frac{L^3}{276EI} \begin{bmatrix} 28 & \frac{18}{L} & -\frac{30}{L} \\ \frac{18}{L} & \frac{51}{L^2} & -\frac{39}{L^2} \\ -\frac{30}{L} & -\frac{39}{L^2} & \frac{111}{L^2} \end{bmatrix} \begin{Bmatrix} -P \\ PL \\ 0 \end{Bmatrix}$$

Using matrix inverse, solve for unknown disp.

Example Problem #1 (cont.)

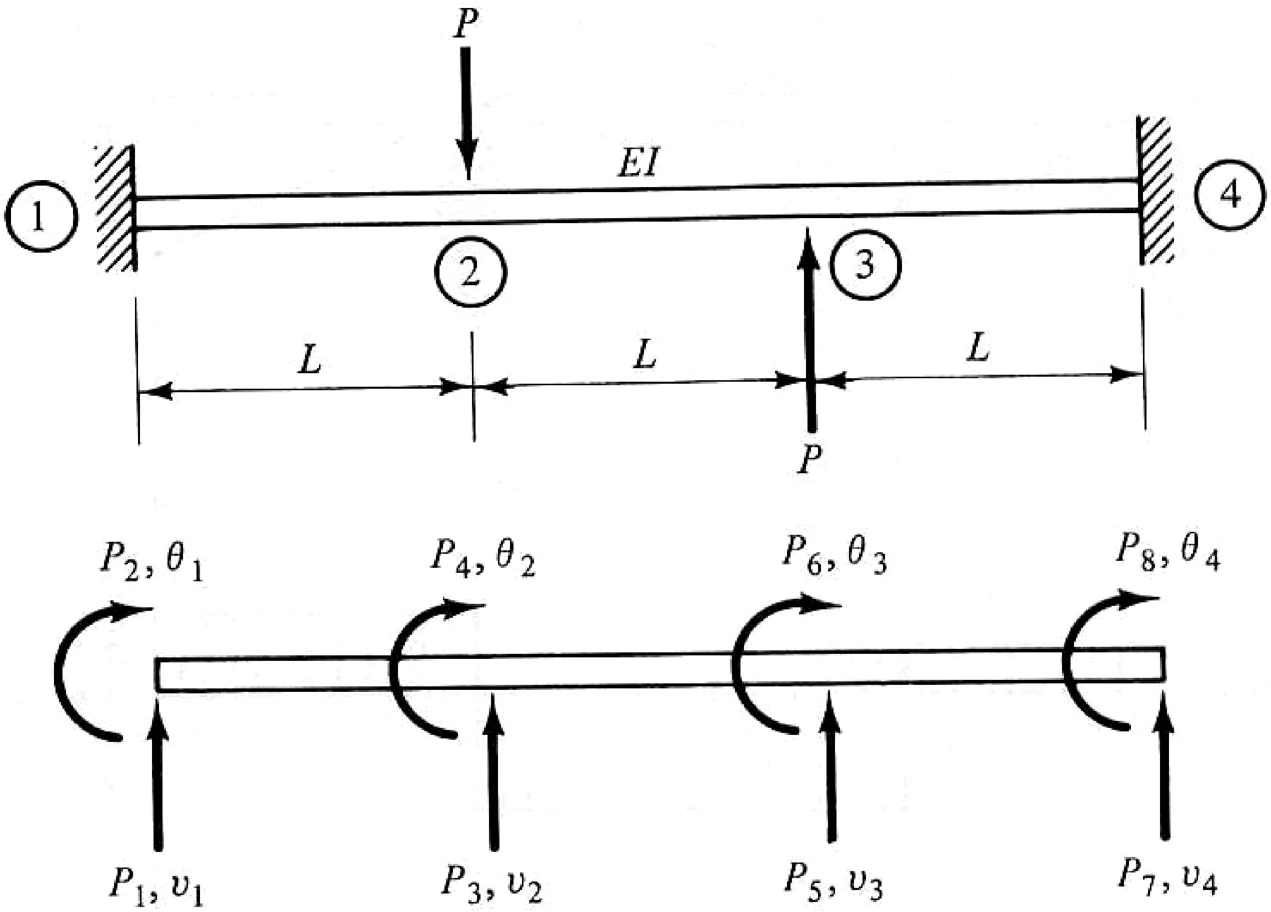
$$\begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \frac{PL^3}{276EI} \begin{Bmatrix} -10 \\ \dfrac{33}{L} \\ -\dfrac{9}{L} \end{Bmatrix}$$

Solve for unknown disp.

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_5 \end{Bmatrix} = \frac{EI}{L} \begin{bmatrix} -\dfrac{12}{L^2} & \dfrac{6}{L} & 0 \\ -\dfrac{6}{L} & 2 & 0 \\ -\dfrac{3}{L^2} & -\dfrac{3}{L} & -\dfrac{3}{L} \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

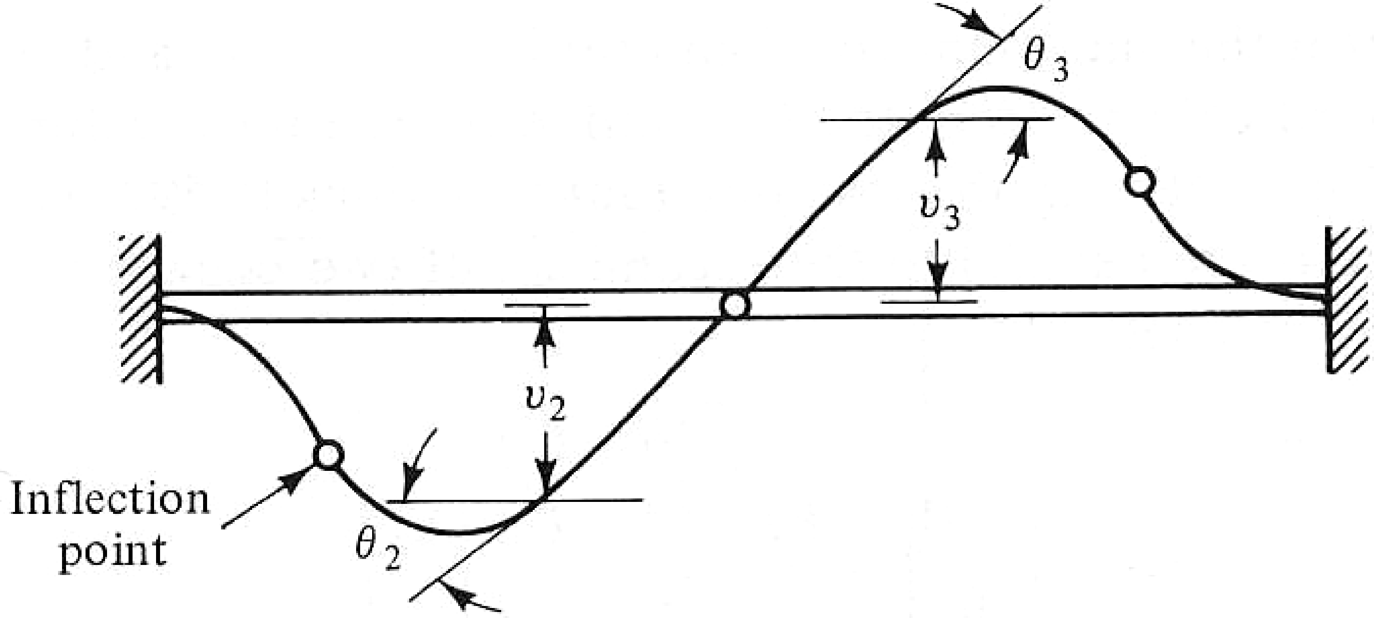
Using disp found above, solve for unknown loads.

Example Problem #2

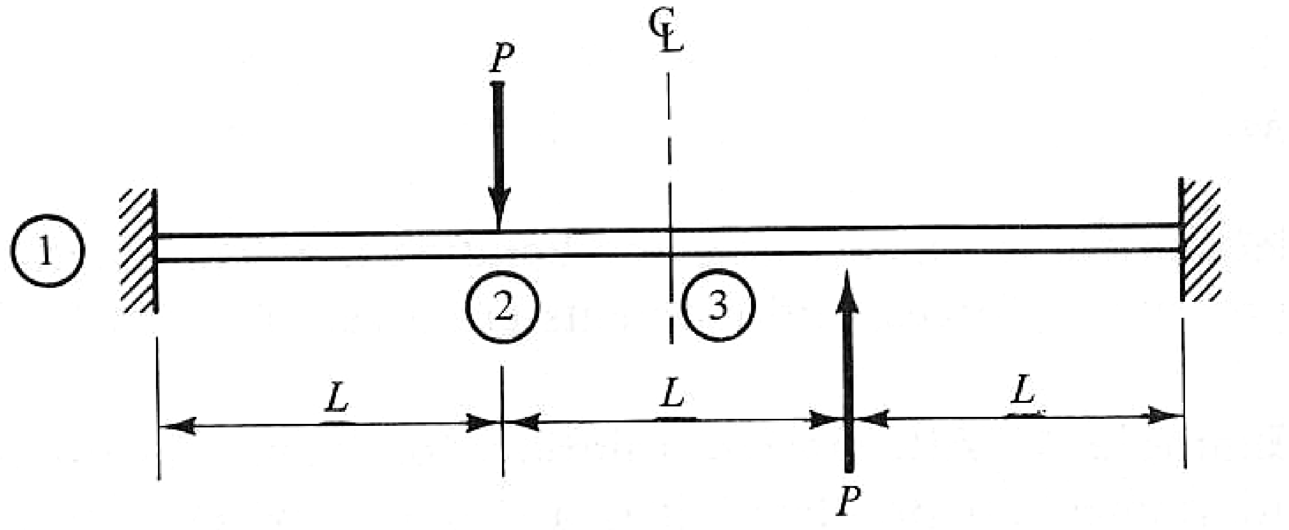


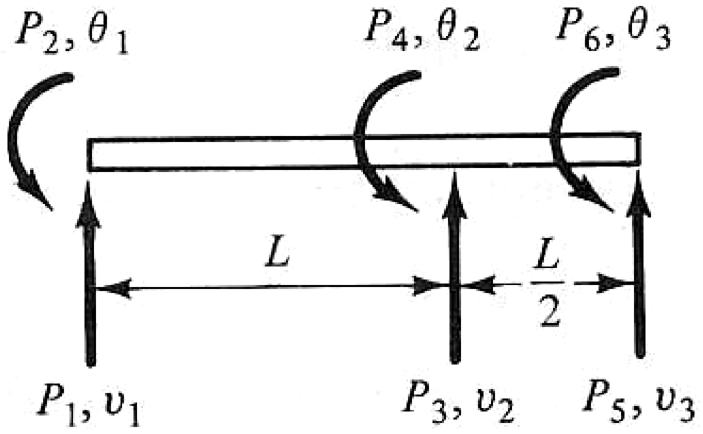
Example Problem #2 (cont.)

Deformed shape of symmetric structure under anti-symmetric loading



Example Problem #2 (cont.)





Example Problem #2 (cont.)

Combining 2 element Stiffness Matrices,

$$\begin{Bmatrix} P_3 \\ P_4 \\ P_6 \end{Bmatrix} = EI \begin{bmatrix} \frac{12}{L^3}+\frac{96}{L^3} & -\frac{6}{L^2}+\frac{24}{L^2} & \frac{24}{L^2} \\ -\frac{6}{L^2}+\frac{24}{L^2} & \frac{4}{L}+\frac{8}{L} & \frac{4}{L} \\ \frac{24}{L^2} & \frac{4}{L} & \frac{8}{L} \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

Example Problem #2 (cont.)

$$\begin{Bmatrix} -P \\ 0 \\ 0 \end{Bmatrix} = \frac{EI}{L} \begin{bmatrix} \frac{108}{L^2} & \frac{18}{L} & \frac{24}{L} \\ \frac{18}{L} & 12 & 4 \\ \frac{24}{L} & 4 & 8 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

Deformations are obtained as follows,

$$\begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \frac{L^3}{2592EI} \begin{bmatrix} 80 & & \\ -\frac{48}{L} & \text{not needed} & \\ -\frac{216}{L} & & \end{bmatrix} \begin{Bmatrix} -P \\ 0 \\ 0 \end{Bmatrix} = \frac{PL^2}{162EI} \begin{Bmatrix} -5L \\ 3 \\ \frac{27}{2} \end{Bmatrix}$$

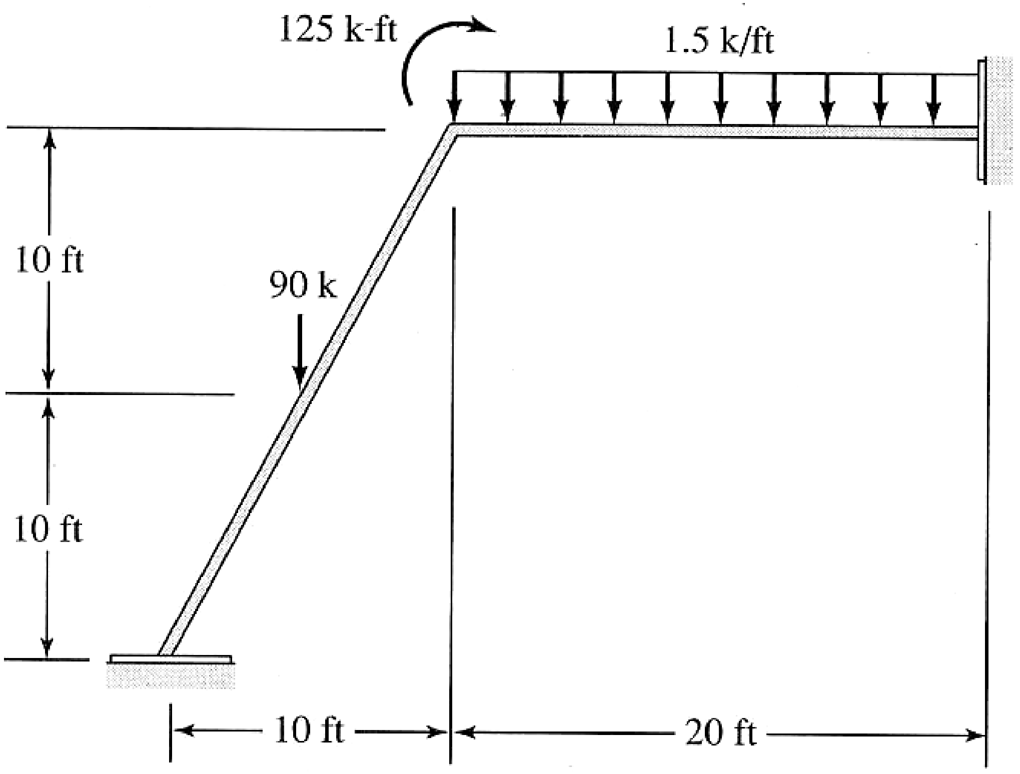
Example Problem #2 (cont.)

Forces and Moments at the Nodal Points (including supports)

$$\begin{Bmatrix} Y_1 \\ M_1 \\ Y_2 \\ M_2 \end{Bmatrix} = \frac{P}{27} \begin{Bmatrix} 13 \\ 6L \\ -13 \\ 7L \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} Y_2 \\ M_2 \\ Y_3 \\ M_3 \end{Bmatrix} = \frac{P}{27} \begin{Bmatrix} -14 \\ -7L \\ 14 \\ 0 \end{Bmatrix}$$

Example Problem

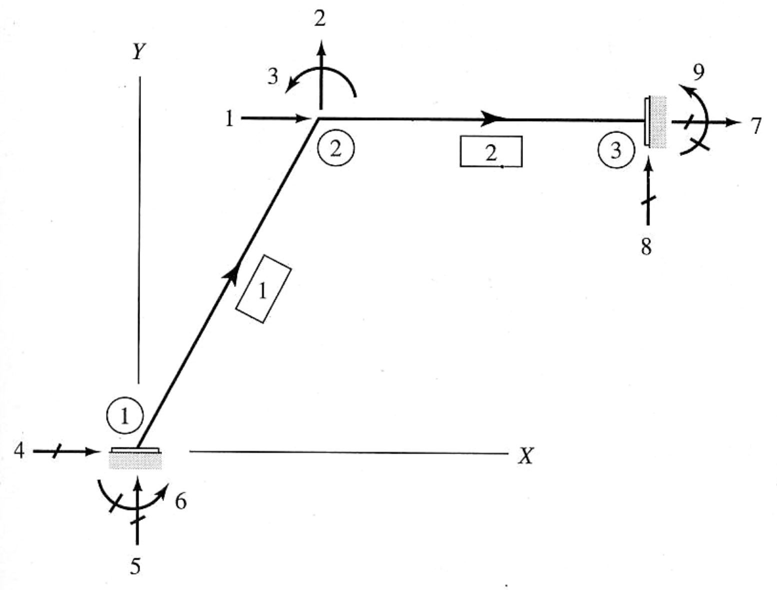
- Determine the joint displacements, member end forces and support reactions for the two member frame.



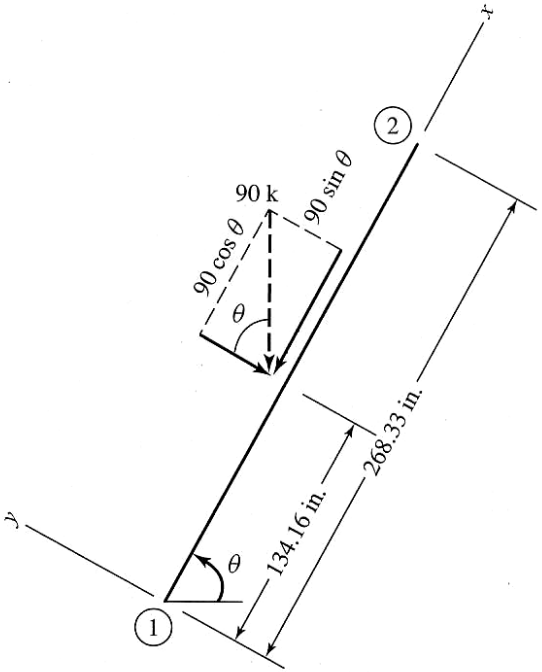
$E, A, I =$ constant
$E = 29{,}000$ ksi
$A = 11.8$ in.^{2}
$I = 310$ in.^{4}

(a) Frame

Example Problem



(b) Analytical Model



(c) Loading on Member 1

Member 1 As shown in Fig. 6.17(b), we have selected joint 1 as the beginning joint, and joint 2 as the end joint for this member. By applying Eqs. (3.62), we determine

$$L = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2} = \sqrt{(10-0)^2 + (20-0)^2}$$

$$= 22.361 \text{ ft} = 268.33 \text{ in.}$$

$$\cos\theta = \frac{X_2 - X_1}{L} = \frac{10-0}{22.361} = 0.44721$$

$$\sin\theta = \frac{Y_2 - Y_1}{L} = \frac{20-0}{22.361} = 0.89443$$

$$\mathbf{K}_1 = \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc:ccc} 259.53 & 507.89 & -670.08 & -259.53 & -507.89 & -670.08 \\ 507.89 & 1{,}021.4 & 335.04 & -507.89 & -1{,}021.4 & 335.04 \\ -670.08 & 335.04 & 134{,}015 & 670.08 & -335.04 & 67{,}008 \\ \hdashline -259.53 & -507.89 & 670.08 & 259.53 & 507.89 & 670.08 \\ -507.89 & -1{,}021.4 & -335.04 & 507.89 & 1{,}021.4 & -335.04 \\ -670.08 & 335.04 & 67{,}008 & 670.08 & -335.04 & 134{,}015 \end{array}\right] \begin{array}{c} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array} \end{array}$$

Fixed End Forces on Member 1

$$W_x = 90 \sin\theta = 90(0.89443) = 80.498 \text{ k}$$

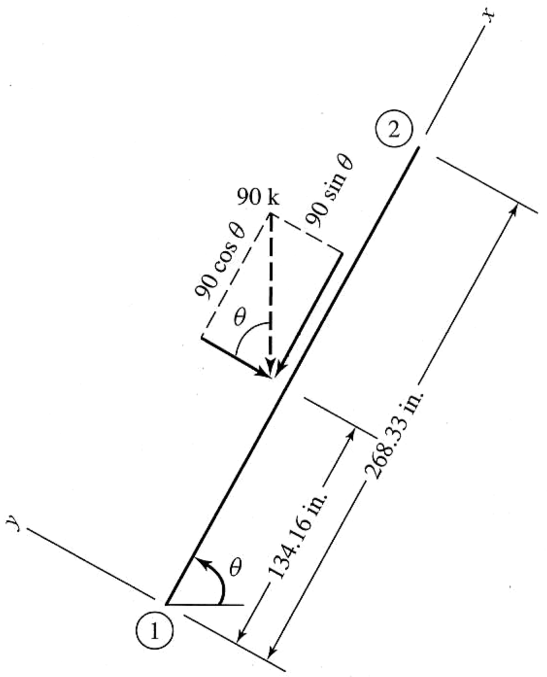
$$W_y = 90 \cos\theta = 90(0.44721) = 40.249 \text{ k}$$

$$FA_b = FA_e = \frac{80.498}{2} = 40.249 \text{ k}$$

$$FS_b = FS_e = \frac{40.249}{2} = 20.125 \text{ k}$$

$$FM_b = -FM_e = \frac{40.249(268.33)}{8} = 1{,}350 \text{ k-in.}$$

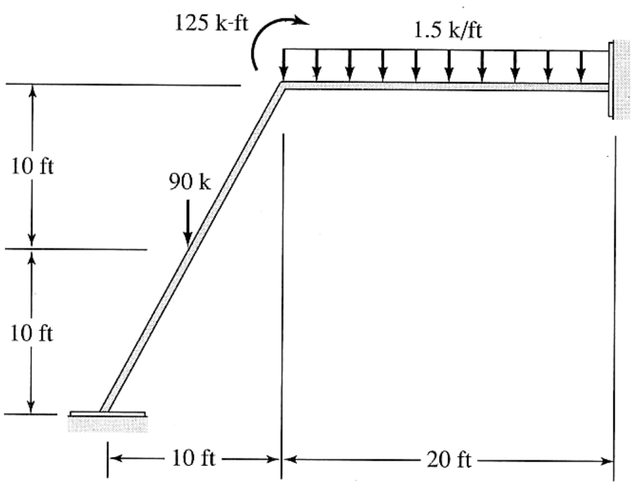
$$\mathbf{F}_{f1} = \left[\begin{array}{r} 0 \\ 45 \\ 1{,}350 \\ \hline 0 \\ 45 \\ -1{,}350 \end{array}\right] \begin{array}{l} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array}$$



(c) Loading on Member 1

Analysis of Member 2

Member 2 As this member is horizontal, with its left-end joint 2 selected as the beginning joint, no coordinate transformations are needed; that is, $\mathbf{T}_2 = \mathbf{I}$, $\mathbf{K}_2 = \mathbf{k}_2$, and $\mathbf{F}_{f2} = \mathbf{Q}_{f2}$. Thus, by substituting $L = 240$ in., $E = 29{,}000$ ksi, $A = 11.8$ in.^{2}, and $I = 310$ in.^{4} into Eq. (6.6), we obtain



$E, A, I =$ constant
$E = 29{,}000$ ksi
$A = 11.8$ in.^{2}
$I = 310$ in.^{4}

(a) Frame

$$\mathbf{K}_2 = \mathbf{k}_2 = \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{cccccc|c} 1 & 2 & 3 & 7 & 8 & 9 & \\ 1{,}425.8 & 0 & 0 & -1{,}425.8 & 0 & 0 & 1 \\ 0 & 7.8038 & 936.46 & 0 & -7.8038 & 936.46 & 2 \\ 0 & 936.46 & 149{,}833 & 0 & -936.46 & 74{,}917 & 3 \\ -1{,}425.8 & 0 & 0 & 1{,}425.8 & 0 & 0 & 7 \\ 0 & -7.8038 & -936.46 & 0 & 7.8038 & -936.46 & 8 \\ 0 & 936.46 & 74{,}917 & 0 & -936.46 & 149{,}833 & 9 \end{array}$$

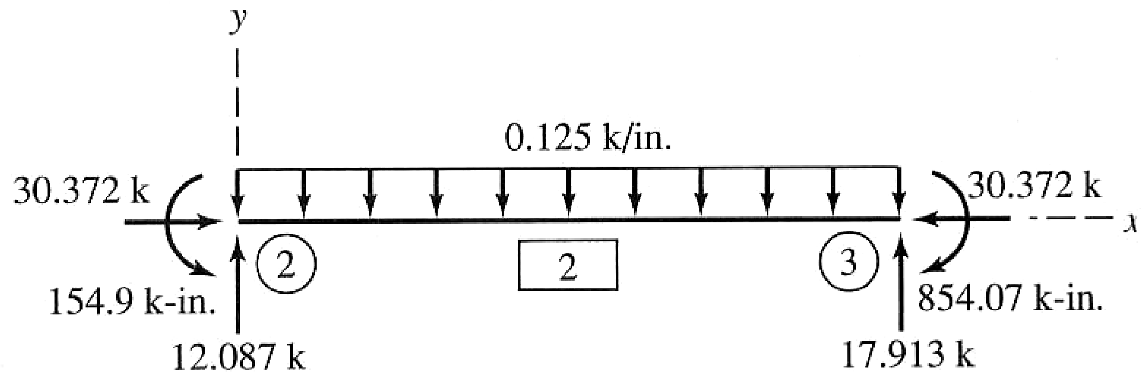
$$FA_b = FA_e = 0$$

$$FS_b = FS_e = \frac{0.125(240)}{2} = 15 \text{ k}$$

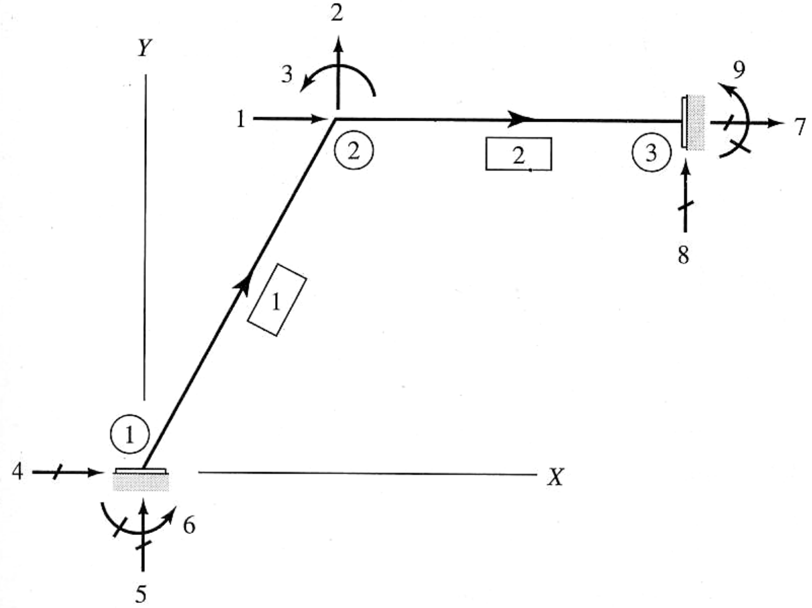
$$FM_b = -FM_e = \frac{0.125(240)^2}{12} = 600 \text{ k-in.}$$

$$\mathbf{F}_{f2} = \mathbf{Q}_{f2} = \begin{array}{c} \left[\begin{array}{c} 0 \\ 15 \\ 600 \\ \hdashline 0 \\ 15 \\ -600 \end{array}\right] \end{array}\begin{array}{c} 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array}$$

Member End Forces in Local Coordinates



Example Problem



(b) Analytical Model

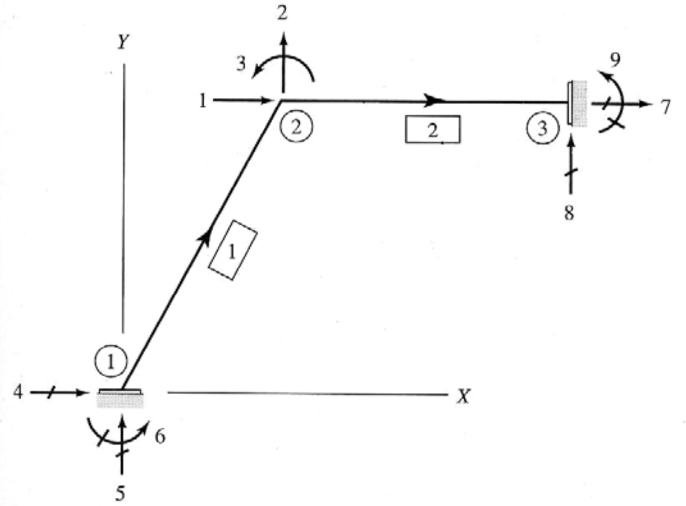
$$\mathbf{S} = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} 259.53 + 1{,}425.8 & 507.89 & 670.08 \\ 507.89 & 1{,}021.4 + 7.8038 & -335.04 + 936.46 \\ 670.08 & -335.04 + 936.46 & 134{,}015 + 149{,}833 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}$$

$$= \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} 1{,}685.3 & 507.89 & 670.08 \\ 507.89 & 1{,}029.2 & 601.42 \\ 670.08 & 601.42 & 283{,}848 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}$$

$$\mathbf{P}_f = \left[\begin{array}{c} 0 \\ 45 + 15 \\ -1{,}350 + 600 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \end{array} = \left[\begin{array}{r} 0 \\ 60 \\ -750 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

(d) Structure Stiffness Matrix and Fixed-Joint Force Vector

Solution of the Assembled Structural System



(b) Analytical Model

$$\mathbf{P} - \mathbf{P}_f = \mathbf{S}\mathbf{d}$$

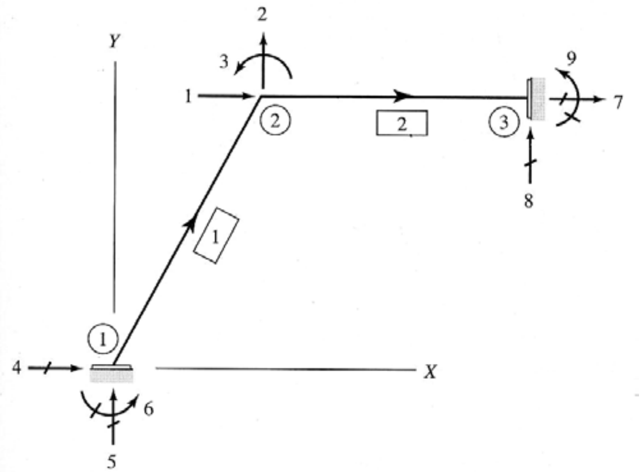
$$\begin{bmatrix} 0 \\ 0 \\ -1{,}500 \end{bmatrix} - \begin{bmatrix} 0 \\ 60 \\ -750 \end{bmatrix} = \begin{bmatrix} 1{,}685.3 & 507.89 & 670.08 \\ 507.89 & 1{,}029.2 & 601.42 \\ 670.08 & 601.42 & 283{,}848 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -60 \\ -750 \end{bmatrix} = \begin{bmatrix} 1{,}685.3 & 507.89 & 670.08 \\ 507.89 & 1{,}029.2 & 601.42 \\ 670.08 & 601.42 & 283{,}848 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\mathbf{P} = \begin{array}{c} \left[\begin{array}{c} 0 \\ 0 \\ -1{,}500 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

$$\mathbf{d} = \begin{bmatrix} 0.021302 \text{ in.} \\ -0.06732 \text{ in.} \\ -0.0025499 \text{ rad} \end{bmatrix}$$

Computing End Forces for Member 1



(b) Analytical Model

$$\mathbf{v}_1 = \begin{array}{c|c} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} & \begin{array}{c} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array} \end{array} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.021302 \text{ in.} \\ -0.06732 \text{ in.} \\ -0.0025499 \text{ rad} \end{bmatrix}$$

$$\mathbf{T}_1 = \begin{bmatrix} 0.44721 & 0.89443 & 0 & 0 & 0 & 0 \\ -0.89443 & 0.44721 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.44721 & 0.89443 & 0 \\ 0 & 0 & 0 & -0.89443 & 0.44721 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

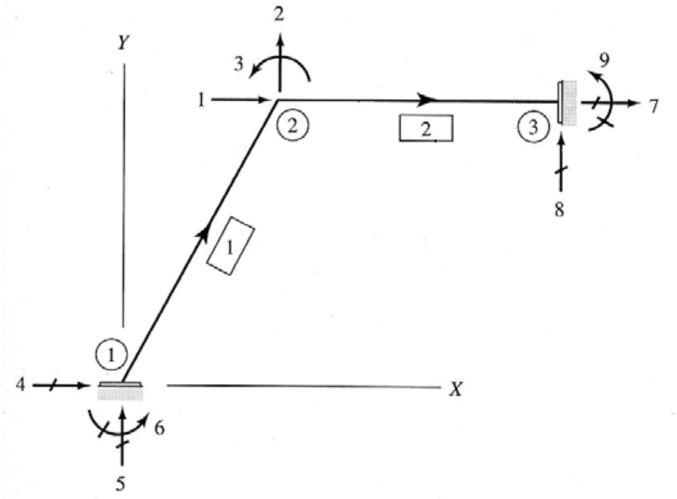
$$\mathbf{d} = \begin{bmatrix} 0.021302 \text{ in.} \\ -0.06732 \text{ in.} \\ -0.0025499 \text{ rad} \end{bmatrix}$$

$$\mathbf{u}_1 = \mathbf{T}_1\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.050686 \text{ in.} \\ -0.04916 \text{ in.} \\ -0.0025499 \text{ rad} \end{bmatrix}$$

Computing End Forces for Member 1

$$\mathbf{k}_1 = \begin{bmatrix} 1{,}275.3 & 0 & 0 & -1{,}275.3 & 0 & 0 \\ 0 & 5.584 & 749.17 & 0 & -5.584 & 749.17 \\ 0 & 749.17 & 134{,}015 & 0 & -749.17 & 67{,}008 \\ -1{,}275.3 & 0 & 0 & 1{,}275.3 & 0 & 0 \\ 0 & -5.584 & -479.17 & 0 & 5.584 & -749.17 \\ 0 & 749.17 & 67{,}008 & 0 & -749.17 & 134{,}015 \end{bmatrix}$$

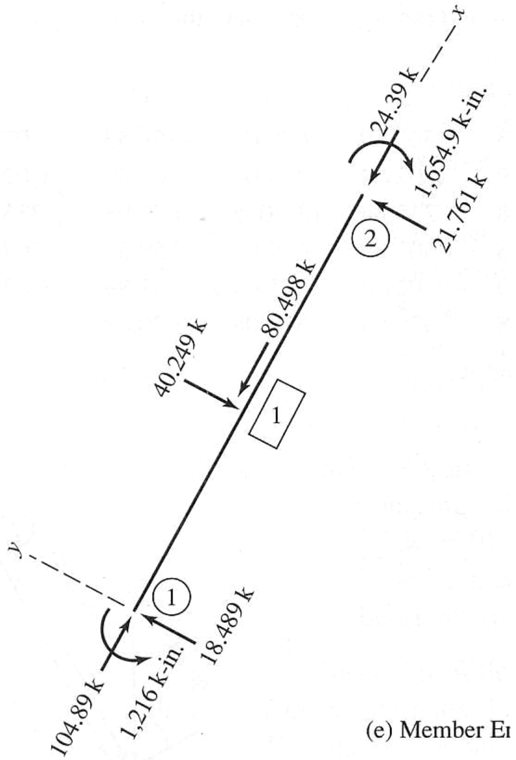
$$\mathbf{Q}_{f1} = \begin{bmatrix} 40.249 \\ 20.125 \\ 1{,}350 \\ 40.249 \\ 20.125 \\ -1{,}350 \end{bmatrix}$$



(b) Analytical Model

$$\mathbf{Q}_1 = \mathbf{k}_1\mathbf{u}_1 + \mathbf{Q}_{f1} = \begin{bmatrix} 104.89 \text{ k} \\ 18.489 \text{ k} \\ 1{,}216 \text{ k-in.} \\ -24.39 \text{ k} \\ 21.761 \text{ k} \\ -1{,}654.9 \text{ k-in.} \end{bmatrix}$$

Equilibrium Check for Member 1



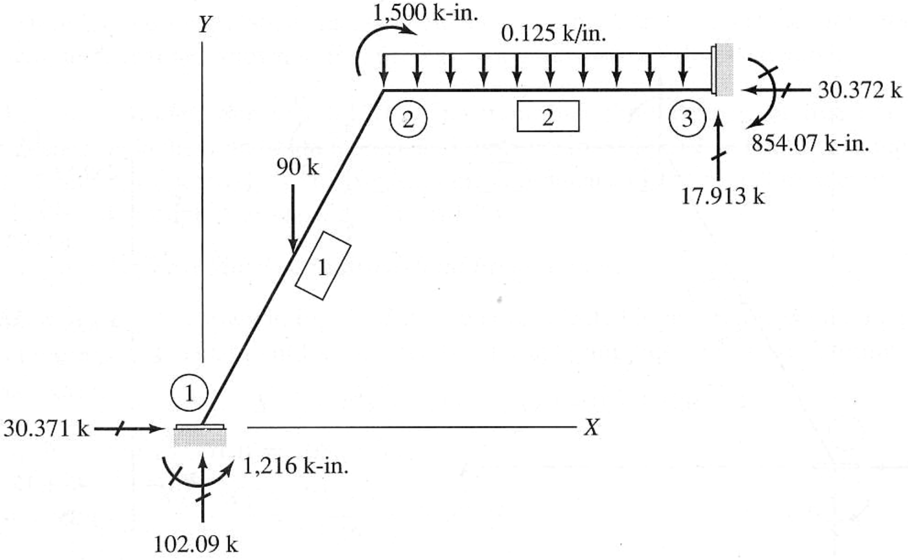
(e) Member E

$$+\nearrow \sum F_x = 0 \qquad 104.89 - 80.498 - 24.39 = 0.002 \cong 0$$

$$+\nwarrow \sum F_y = 0 \qquad 18.489 - 40.249 + 21.761 = 0.001 \cong 0$$

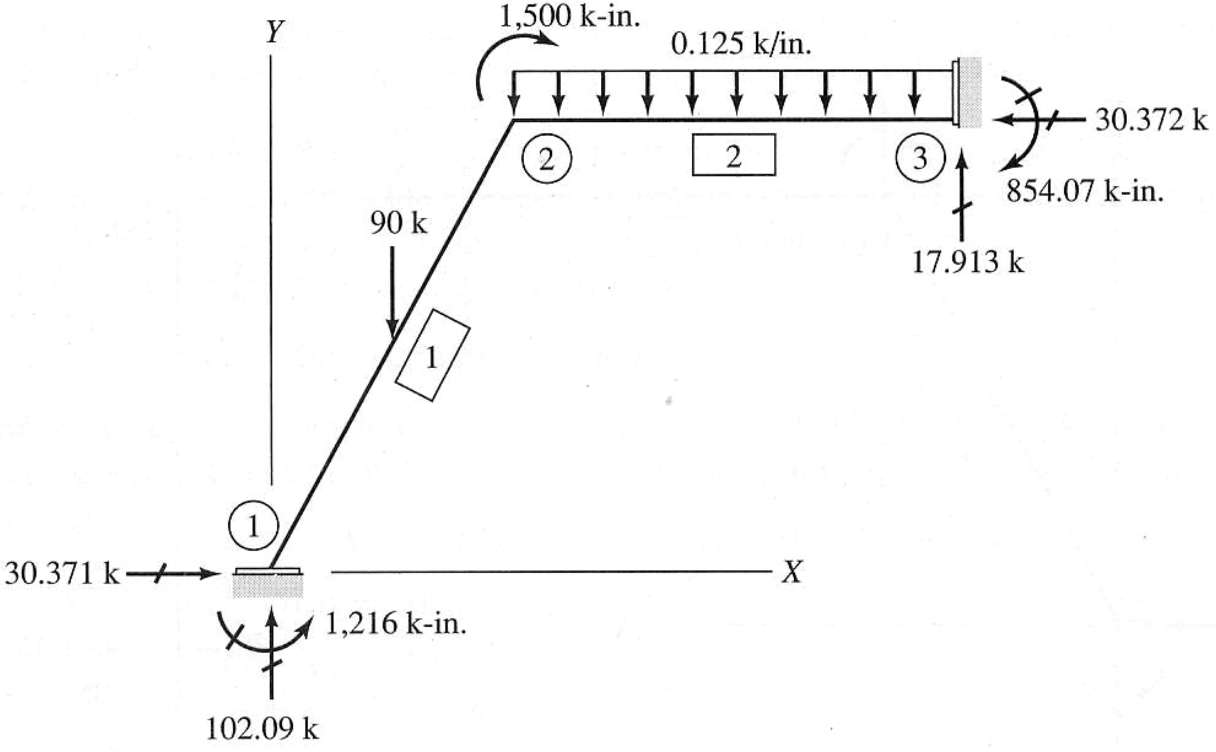
$$+\circlearrowleft \sum M_{②} = 0 \qquad 1{,}216 - 18.489(268.33) + 40.249(134.16) - 1{,}654.9$$
$$= -0.25 \cong 0$$

$$\mathbf{F}_1 = \mathbf{T}_1^T \mathbf{Q}_1 = \begin{array}{c} \left[\begin{array}{r} 30.371 \\ 102.09 \\ 1{,}216 \\ \hdashline -30.371 \\ -12.083 \\ -1{,}654.9 \end{array}\right] \end{array} \begin{array}{c} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array}$$



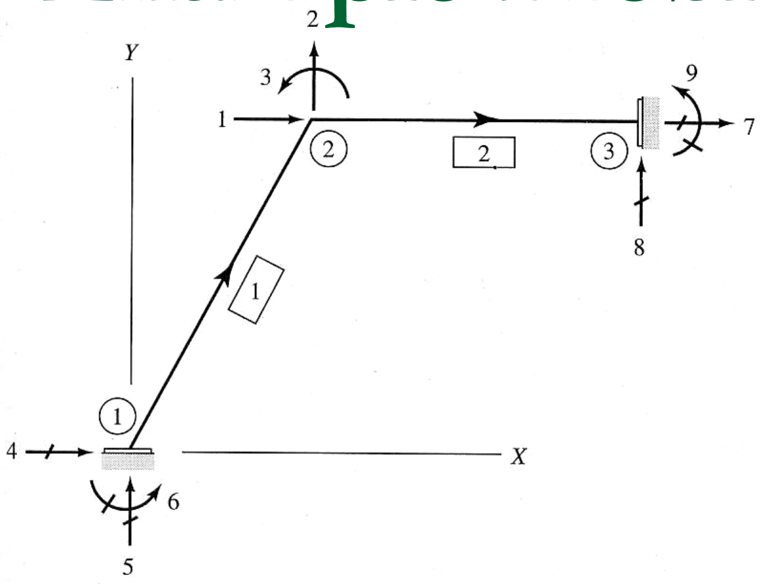
(g) Support Reactions

Example Problem



(g) Support Reactions

Example Problem



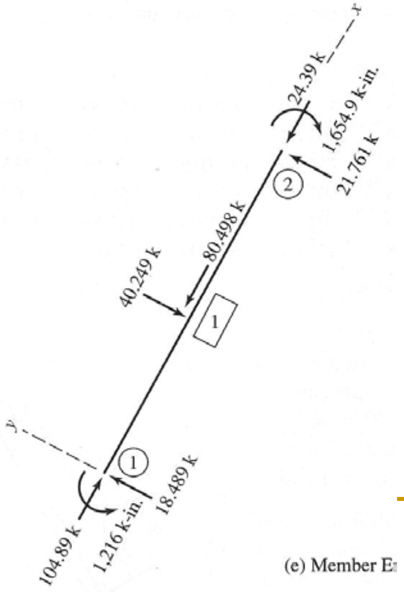
(b) Analytical Model

$$\mathbf{S} = \begin{array}{cccc} 1 & 2 & 3 & \\ 259.53 + 1{,}425.8 & 507.89 & 670.08 & 1 \\ 507.89 & 1{,}021.4 + 7.8038 & -335.04 + 936.46 & 2 \\ 670.08 & -335.04 + 936.46 & 134{,}015 + 149{,}833 & 3 \end{array}$$

$$= \begin{array}{cccc} 1 & 2 & 3 & \\ 1{,}685.3 & 507.89 & 670.08 & 1 \\ 507.89 & 1{,}029.2 & 601.42 & 2 \\ 670.08 & 601.42 & 283{,}848 & 3 \end{array}$$

$$\mathbf{P}_f = \left[\begin{array}{c} 0 \\ 45 + 15 \\ -1{,}350 + 600 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \end{array} = \left[\begin{array}{r} 0 \\ 60 \\ -750 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

(d) Structure Stiffness Matrix and Fixed-Joint Force Vector



(e) Member E

$$\mathbf{R} = \left[\begin{array}{r} 30.371 \text{ k} \\ 102.09 \text{ k} \\ 1,216 \text{ k-in.} \\ -30.372 \text{ k} \\ 17.913 \text{ k} \\ -854.07 \text{ k-in.} \end{array}\right] \begin{array}{c} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}$$

(f) Support Reaction Vector